

Anyon statistics with continuous variablesJing Zhang,^{1,*} Changde Xie,¹ Kunchi Peng,¹ and Peter van Loock²¹*State Key Laboratory of Quantum Optics and Quantum Optics Devices, Institute of Opto-Electronics, Shanxi University, Taiyuan 030006, People's Republic of China*²*Optical Quantum Information Theory Group, Institute of Theoretical Physics I and Max-Planck Research Group, Institute of Optics, Information and Photonics, Universität Erlangen-Nürnberg, Staudtstrasse 7/B2, 91058 Erlangen, Germany*

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We describe a continuous-variable scheme for simulating the Kitaev lattice model and for detecting statistics of Abelian anyons. The corresponding quantum optical implementation is solely based upon Gaussian resource states and Gaussian operations, hence allowing for a highly efficient creation, manipulation, and detection of anyons. This approach extends our understanding of the control and application of anyons and it leads to the possibility for experimental proof-of-principle demonstrations of anyonic statistics using continuous-variable systems.

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I. INTRODUCTION

It is well known from statistical physics that in three spatial dimensions, only two kinds of particles exist, bosons and fermions. A double exchange of any of these particles, topologically equivalent to braiding one around the other, will have no effect in three dimensions. In two spatial dimensions, however, braiding one particle around the other is no longer equivalent to doing nothing. Hence, the quantum mechanical wave function can acquire any phase through the braiding operation, leading to the fractional statistics of exotic quasiparticles, so-called anyons [1]. Such anyons may be realized in model systems that have highly entangled ground states with topological order [2–4]. The investigation of anyonic fractional statistics is not only of fundamental interest. It also led to the new field of topological quantum computation, where the intrinsic robustness of the ground-state Hamiltonian against local perturbations offers naturally given fault tolerance for quantum-information processing [5–8].

The concept of anyons first emerged in connection with the fractional quantum Hall effect [2], which occurs in a two-dimensional electron gas at low temperatures being subject to a perpendicular magnetic field. The observation of anyonic features in these systems requires cooling to very low temperatures where only the ground and a few excited states are populated. At the same time, anyons emerge as spatially localized many-body states that can be created and transported through the many-body system via local operations. Although some signatures have been observed in the fractional quantum Hall systems [9], a direct observation of fractional statistics associated with anyon braiding has not been achieved there. Nonetheless, various theoretical proposals exist [10], including a spin-model scheme based upon laser manipulation of cold atoms in an optical lattice [11]. However, this scheme appears to be beyond current experimental capabilities.

A very simple model Hamiltonian from which one can obtain Abelian anyons was proposed by Kitaev for spin- $\frac{1}{2}$ (qubit) systems [5]. This surface code model involves suitable combinations of four-body interactions. As such interactions are still hard to achieve on the level of the ground-state Hamiltonian, a more practical approach to simulating the surface code model would be based upon the creation and manipulation of multiparty entangled graph states [12]. Instead of cooling the system to its ground state, the ground and excited states are created dynamically from graph states. By focussing on the underlying multiparty entangled states, this approach provides a testbed for demonstrating anyonic properties independent of the presence of a Hamiltonian, in a simpler and more efficient way compared to the traditional approaches based on complex solid-state systems (though in order to obtain the full robustness needed for fault-tolerant topological quantum computation, the background Hamiltonian defining the protected code space appears to be crucial).

The graph-state-based protocol was demonstrated experimentally with six-photon [13] and four-photon graph states [14]. As reflected by these experiments, both theoretical and experimental research of anyonic fractional statistics was mainly concentrated on two-level or spin- $\frac{1}{2}$ systems. Here, we shall consider anyonic statistics in continuous-variable (CV) systems [15], i.e., for quantized harmonic oscillators representing, in particular, quantized optical modes. In order to apply the graph-state-based approach [12] to CV systems, we shall consider CV cluster and graph states [16]. The corresponding optical Gaussian states can be efficiently and unconditionally generated using off-line squeezed-state resources and linear optics [16,17]. Various four-mode CV cluster states have been experimentally realized already [18,19]. By adding a non-Gaussian measurement to the toolbox of Gaussian operations, i.e., homodyne measurements within the model of measurement-based quantum computation [20], universal quantum computation is, in principle, possible using CV Gaussian cluster states [21].

Here, within the CV setting, we will show that the anyonic ground state (or surface code state) can be created using Gaussian resource states (off-line squeezed states) and

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linear optics. Generating quasiparticle excitations and implementing the braiding and fusion operations only requires simple single-mode Gaussian operations; also detection and verification of the quasiparticle statistics (observation of the fractional phase) can be achieved via Gaussian operations including homodyne detection.

The plan of the paper is as follows. First, in Sec. II, we will briefly introduce the formalism and notations used in the paper for describing CV quantum logic and computation. In the following, we discuss a generation scheme for the CV anyonic ground state based on CV graph states (Sec. III) and how to achieve the anyonic excitations including the braiding operations (Sec. IV). In Sec. V, we will describe a detection scheme to verify the fractional phase of the anyons; in this measurement scheme, the global phase acquired by the excited state through the braiding operation is turned into a phase-space translation resulting in different output stabilizer operators (corresponding to different position-momentum linear combinations) depending on whether a braiding loop has been applied or not. Finally, before the conclusion in Sec. VII, we discuss in more detail potential optical implementations of our CV-based anyon realization (Sec. VI).

II. CONTINUOUS-VARIABLE COMPUTATION

In order to consider quantum logic over continuous variables, the qubit Pauli X and Z operators are generalized to the Weyl-Heisenberg (WH) group [22], which is the group of phase-space displacements. This is a Lie group with generators $\hat{x}=(\hat{a}+\hat{a}^\dagger)/\sqrt{2}$ (quadrature-amplitude or position) and $\hat{p}=i(\hat{a}-\hat{a}^\dagger)/\sqrt{2}$ (quadrature-phase or momentum) representing, for instance, a single quantized mode (qumode) of the electromagnetic field. These operators satisfy the canonical commutation relation $[\hat{x}, \hat{p}]=i$ (with $\hbar=1$). In analogy to the qubit Pauli operators, the single-mode WH operators are defined as $X(s)=\exp(-is\hat{p})$ and $Z(t)=\exp(it\hat{x})$ with $s, t \in \mathbb{R}$. The WH operator $X(s)$ is a position-translation operator, which acts on the computational basis of position eigenstates $\{|x\rangle; x \in \mathbb{R}\}$ as $X(s)|x\rangle=|x+s\rangle$; $Z(t)$ is a momentum-translation operator, which acts on the momentum eigenstates as $Z(t)|p\rangle=|p+t\rangle$. These operators are noncommutative and obey the identity

$$X(s)Z(t) = e^{-ist}Z(t)X(s). \tag{1}$$

In the following, in analogy to the qubit case, we denote the WH operators as Pauli operators. The Pauli operators for one mode can be used to construct a set of Pauli operators $\{X_i(s_i), Z_i(t_i); i=1, \dots, n\}$ for n -mode systems. This set generates the Pauli group \mathcal{C}_1 . The Clifford group \mathcal{C}_2 is the group of transformations that preserve the Pauli group \mathcal{C}_1 under conjugation; i.e., a gate U is an element of the Clifford group if $URU^\dagger \in \mathcal{C}_1$ for every $R \in \mathcal{C}_1$. The Clifford group \mathcal{C}_2 for continuous variables corresponds to the (semidirect) product of the Pauli group and the linear symplectic group of all one-mode and two-mode squeezing transformations [22]. The CV gate for switching between the position and the momentum basis is given by the Fourier transform operator $F = \exp[i(\pi/4)(\hat{x}^2 + \hat{p}^2)]$, with $F|x\rangle_x = |x\rangle_p$. This gate corresponds to a generalization of the Hadamard gate for qubits.

The phase gate $P(\eta) = \exp[i(\eta/2)\hat{x}^2]$ is a CV squeezing operation, in analogy to a Clifford phase gate for qubits [23]. The qubit controlled phase (sign) gate is generalized to a controlled- Z , $C_Z = \exp(i\hat{x}_1 \otimes \hat{x}_2)$. This gate provides the basic interaction for two modes 1 and 2; at the same time, it describes a quadrature quantum nondemolition (QND) interaction. The set $\{X(s), F, P(\eta), C_Z; s, \eta \in \mathbb{R}\}$ generates the Clifford group. Transformations within the Clifford group do not form a universal set of gates for CV quantum computation. However, Clifford group transformations (corresponding to Gaussian transformations mapping Gaussian states onto Gaussian states) together with any higher-order nonlinear transformation acting on a single mode (corresponding to a non-Gaussian transformation with a Hamiltonian at least cubic in the mode operators) form a universal set of gates [22,24]. As we will show in the following, all operations needed to demonstrate the anyonic fractional statistics in CV systems are entirely based upon Clifford group (Gaussian) transformations and homodyne measurements (Gaussian operations).

III. CONTINUOUS-VARIABLE GRAPH AND SURFACE CODE STATES

A graph quantum state is described by a mathematical graph, i.e., a set of vertices connected by edges [25]. A vertex represents a physical system, e.g., a qubit (discrete two-level system in a two-dimensional Hilbert space) or a CV qumode (system with a continuous spectrum in an infinite-dimensional Hilbert space). An edge between two vertices represents the physical interaction between the corresponding systems. The preparation procedure of CV cluster states [16] is analogous to that for qubit cluster states: First, prepare each mode (or graph vertex) in a phase-squeezed state, approximating a zero-momentum eigenstate (the analogue of the Pauli- $X+1$ eigenstate); then apply a QND interaction [the gate $C_Z = \exp(i\hat{x}_j \otimes \hat{x}_k)$] to each pair of modes (j, k) linked by an edge in the graph. Note that all C_Z gates commute. The resulting CV cluster-graph state (and, more generally, any CV cluster-type state [17]) satisfies, in the limit of infinite squeezing [16],

$$\hat{g}_a = \left(\hat{p}_a - \sum_{b \in N_a} \hat{x}_b \right) \rightarrow 0, \quad \forall a \in G, \tag{2}$$

where the modes $a \in G$ correspond to the vertices of the graph of n modes and the modes $b \in N_a$ are the nearest neighbors of mode a . More precisely, this relation describes a simultaneous zero eigenstate of the n position-momentum linear combination operators \hat{g}_a .

The stabilizer operators $G_a(\xi) = \exp(-i\xi\hat{g}_a) = X_a(\xi)\prod_{b \in N_a} Z_b(\xi)$ with $\xi \in \mathbb{R}$ for CV cluster states are analogous to the n independent stabilizers $G_a = X_a \prod_{b \in N_a} Z_b$ for qubit cluster states [25]. Starting from a two-dimensional (2D) CV cluster state, single-mode measurements of every second mode in either the position or the momentum basis, with a subsequent Fourier transform F for all the remaining modes, yields a new graph state $|\psi\rangle$. The measurement pattern is illustrated in Fig. 1(a). An x measurement removes the measured mode from the cluster and breaks all connections be-

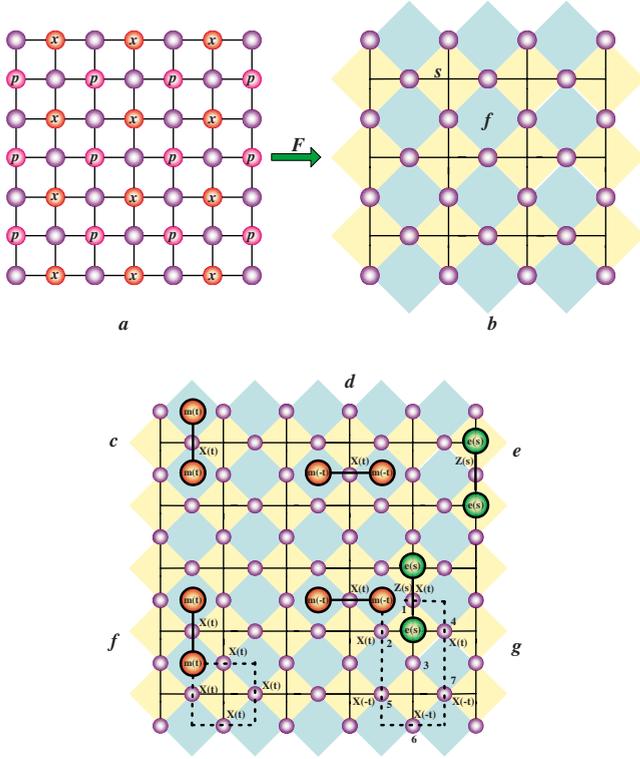


FIG. 1. (Color online) Surface code model and anyonic braiding with continuous variables. (a) The measurement pattern to prepare the anyonic ground state $|\psi\rangle$ from a two-dimensional CV cluster state, where x and p denote a single-mode measurement in the position and momentum basis, respectively, F is the Fourier transform for all remaining modes. (b) The CV surface code model consists of the star operators $A_s(\xi)$ and plaquette operators $B_f(\eta)$. (c) Application of an $X(t)$ position translation on a single mode in the horizontal edge yields a pair of $m(t)$ -type anyons placed at the neighboring plaquettes. (d) Application of $X(t)$ on a single mode in the vertical edge yields a pair of $m(-t)$ -type anyons. (e) Application of $Z(s)$ on a single mode yields a pair of $e(s)$ -type anyons at the neighboring stars. (f) Application of $X(t)$ on the modes along a string would create m -type anyons, where two m -type anyons are created on the same plaquettes. These two anyons annihilate each other according to the fusion rule $m(t) \times m(-t) = 1$. Thus, the two single strings are glued together with two anyons at the endpoints. When a part of a string forms a loop, it cancels (dashed). (g) An m -type anyon follows a closed loop around an e -type anyon.

tween this mode and the rest of the cluster, including momentum-displacements of the nearest modes with the measured result. Conversely, the effect of a p -measurement is to remove the measured mode, but to connect the neighboring modes at the same time. These measurements transform the set of correlations in terms of position-momentum linear combination operators from $[(\hat{p}_a - \sum_{b \in N_a} \hat{x}_b) \rightarrow 0]$ to $[\hat{a}_s = (\hat{p}_{s,1} + \hat{p}_{s,2} + \hat{p}_{s,3} + \hat{p}_{s,4}) \rightarrow 0, \hat{b}_f = (\hat{x}_{f,1} - \hat{x}_{f,2} + \hat{x}_{f,3} - \hat{x}_{f,4}) \rightarrow 0]$ for the remaining modes, where s and f label stars and plaquettes (faces), respectively, and the indices $1, \dots, 4$ of the position and momentum operators denote those modes located at a common star or at the boundary of a common plaquette. The stabilizers $\{A_s(\xi) = \exp(-i\xi\hat{a}_s)$

$= \prod_{j \in \text{star}(s)} X_{s,j}(\xi), B_f(\eta) = \exp(-i\eta\hat{b}_f) = \prod_{j \in \text{boundary}(f)} Z_{f,j}[(-1)^j \eta]\}$, with $\xi, \eta \in \mathbb{R}$, for the new CV state $|\psi\rangle$ are exactly analogous to those of the first Kitaev model [5] for a two-dimensional spin lattice. These stabilizers, the star operators $A_s(\xi)$ and plaquette operators $B_f(\eta)$, all commute. So the new CV state $|\psi\rangle$ corresponds to the anyonic ground state with $A_s(\xi)|\psi\rangle = |\psi\rangle$ and $B_f(\eta)|\psi\rangle = |\psi\rangle, \forall \xi, \eta \in \mathbb{R}$, for all stars s and plaquettes f , in the limit of infinite squeezing. After defining the CV anyonic ground state and describing how to create an approximate version of this state (asymptotically perfect in the limit of infinite squeezing) from CV cluster states, let us now consider anyonic excitations from the ground state and the basic braiding operations.

IV. ANYON CREATION AND BRAIDING

Starting from the ground state one can excite pairs of anyons connected by a string using single-mode operations. Unlike the Pauli group for qubits, the WH group is a continuous group, and therefore quasiparticle excitation is continuous. More specifically, by applying the position-translation operator $X(t)$ to some mode of the lattice, a pair of so-called m -type anyons in the state $|m[(-1)^d t]\rangle = X(t)|\psi\rangle$ ($d \in \{1, 2\}$, $d=1$ means that the relevant mode lies on the vertical edges, whereas $d=2$ refers to the horizontal edges) is created on the two neighboring plaquettes [Figs. 1(c) and 1(d)]. An e pair of anyons, given by $|e(s)\rangle = Z(s)|\psi\rangle$, is obtained on two neighboring stars by a momentum-translation operation $Z(s)$ [Fig. 1(e)].

Similar to the qubit case, the excited states differ from the ground states in their stabilizer operators. In the CV case, this means that some of the position-momentum linear combination operators \hat{a}_s and \hat{b}_f [from the ground-state stabilizers $A_s(\xi) = \exp(-i\xi\hat{a}_s)$ and $B_f(\eta) = \exp(-i\eta\hat{b}_f)$] no longer satisfy the ground-state conditions $\hat{a}_s = 0$ and $\hat{b}_f = 0$ (in the limit of infinite squeezing), but rather become $\hat{a}_s = t$ for $Z_{j \in \text{star}(s)}(t)|\psi\rangle$ and $\hat{b}_f = \pm s$ for $X_{j \in \text{boundary}(f)}(s)|\psi\rangle$. In other words, the anyonic excitations become manifest in “violations” of the stabilizer (“nullifier”) conditions for \hat{a}_s in the presence of e particles and for \hat{b}_f in the presence of m particles.

The fusion rules $[e(s) \times e(t) = e(s+t), m(s) \times m(t) = m(s+t), e(0) = m(0) = 1, 1 \times e(s) = e(s), 1 \times m(s) = m(s)]$ describe the outcome from combining two anyons. Thereby, excited states can be obtained from the CV ground state $|\psi\rangle$ by applying open string operators, which create quasiparticles at their end points [Fig. 1(f)]. The closed X strings surround the corresponding star (plaquette) operators [Fig. 1(f)]. Thus, for any closed X (Z) string, the system will remain in the ground state.

The anyonic character of the excited states is now revealed through a nontrivial phase factor acquired by the wave function of the lattice system after braiding anyons, i.e., after moving m around e [Fig. 1(g)] or vice versa. Consider the initial state $|\Psi_{\text{ini}}\rangle = Z_1(s)|\psi\rangle = |e(s)\rangle$. If an anyon of type m is assumed to be at a neighboring plaquette, it can be moved around e along the path generated by successive

applications of $X(t)$ on the four modes of the star. The final state is

$$\begin{aligned} |\Psi_{\text{fin}}\rangle &= X_1(t)X_2(t)X_3(t)X_4(t)|\Psi_{\text{ini}}\rangle \\ &= e^{-ist}Z_1(s)[X_1(t)X_2(t)X_3(t)X_4(t)|\psi\rangle] \\ &= e^{-ist}|\Psi_{\text{ini}}\rangle, \end{aligned} \quad (3)$$

using Eq. (1). The extra factor $\exp(-ist)$ is the topological phase factor, which reveals the presence of the enclosed anyons (it would not be obtained, if the initial state had been the unexcited state, $|\Psi_{\text{ini}}\rangle=|\psi\rangle$). Note that here in the CV setting, the quantum state can acquire any phase, in contrast to the discrete value for anyons in qubit systems with the analogous braiding operation (where $XZ=-ZX$ for qubits).

The corresponding phase factors, changing the phase of the wave function, constitute an Abelian (commutative) representation of the braid group. It is important to understand that, in general, a relation as described by Eq. (3) can be obtained for different paths of the loop. For instance, the loop 1-2-5-6-7-4 of m -type anyons [Fig. 1(g)], which corresponds to the application of $X_1(t)X_2(t)X_5(-t)X_6(-t)X_7(-t)X_4(t)=[X_1(t)X_2(t)X_3(t)X_4(t)][X_3(-t)X_5(-t)X_6(-t)X_7(-t)]$, involving two star operators, gives the same result for the topological phase as in Eq. (3). In fact, this equivalence reflects the topological character of the extra phase factor and the potential robustness of the corresponding state transformation, as it is exploited in topological approaches to fault-tolerant quantum computation.

V. ANYON DETECTION

In order to detect the most significant feature of anyons, their nontrivial statistical phase obtainable through braiding, we may employ an ‘‘interference measurement’’ in phase space, which enables us to detect the otherwise invisible global phase factor of Eq. (3). The idea is to turn the global phase acquired by the excited state into a displacement in phase space that would not occur without braiding operation.

First, we generate a superposition of the ground state $|\psi\rangle$ and the continuously excited state of the anyon e via the squeezing operation $P(s)=\exp[i(s/2)\hat{x}^2]$ (the CV quadratic phase gate) instead of the momentum-translation operation $Z(s)=\exp(is\hat{x})$. Then we perform a closed loop of anyon m around e . Finally, application of the inverse squeezing operation $P(-s)=\exp[-i(s/2)\hat{x}^2]$ makes the phase difference visible, leading to the state $e^{-it^2s/2}|e(st)\rangle$ according to

$$e^{-i(s/2)\hat{x}^2}e^{-it\hat{p}}e^{i(s/2)\hat{x}^2}=e^{-it^2s/2}e^{ist\hat{x}}e^{-it\hat{p}}. \quad (4)$$

This compares to the state $e^{-i(s/2)\hat{x}^2}e^{i(s/2)\hat{x}^2}|\psi\rangle=|\psi\rangle$ without a closed loop of anyon m around e . In other words, the effect of the braiding operation can be measured via the different stabilizer operators or, more precisely, the different position-momentum linear combination operators of the output states: For the relevant star operator s_0 , we have $\hat{a}_{s_0}=st$ with braiding and $\hat{a}_{s_0}=0$ without braiding.

Note that this verification scheme is only complete, provided that, in addition to detecting the change of the stabilizer (nullifier) operator \hat{a}_{s_0} , a sufficient set of extra nullifiers

\hat{a}_s and \hat{b}_f is measured in order to confirm the full inseparability of the output states (the ground state and the momentum-displaced ground state) [26]. Otherwise, even a fully separable product state of zero-momentum eigenstates used as ‘‘ground state’’ could produce the same braiding statistics, as it would satisfy $\hat{a}_{s_0}=0$ without braiding and $\hat{a}_{s_0}=st$ with a ‘‘closed braiding loop.’’ Such a trivial ground state, however, would obviously not simulate Kitaev’s surface code model; in particular, apart from not satisfying the complete set of ground-state correlations, it would not allow for a verification of excited m particles or the ‘‘conjugate’’ braiding operation of moving e around m .

VI. OPTICAL IMPLEMENTATION

In a real experiment, because of finite squeezing, it is impossible to create perfect correlations in terms of position-momentum linear combination operators (the stabilizers or, better, nullifiers) for the anyonic ground state $|\psi\rangle$. Nonetheless, we may still implement and detect anyonic fractional statistics by interpreting the presence of small fluctuations around the ground-state stabilizer conditions due to finite squeezing, without the existence of any first-moment phase-space translations, as the ground state [Fig. 2(d)]; sufficiently large first-moment position (or momentum) translations, still maintaining the second-moment multimode position-momentum correlations, would then correspond to the excited states.

For verification, the amplitude (phase) quadrature, i.e., the position (momentum) of each mode is measured via homodyne detection and every detector output is sampled within a time interval to yield a number of measured quadrature values. Using the correlation diagram [27] containing a sufficient number of measured quadrature values, one can determine first and second moments of the position-momentum linear combination operators for the anyonic ground and excited states. Thus, the fully inseparable ground states can be verified and discriminated from the (for large squeezing near-orthogonal) excited (displaced ground) states [26]; at the same time enabling one to confirm the braiding-dependent state transition according to Eq. (4) [see Fig. 2(d)]. For finite, even small squeezing values, it is still possible to discriminate the ground and excited states, provided the amplitude (or phase) translations are sufficiently large. Higher squeezing values improve the sensitivity for detecting the amplitude (or phase) translations of the excited states. Note that even for zero-squeezing resources, such a state discrimination can be almost perfect (for large amplitudes and/or phases); in this case, more subtle is the verification of the full inseparability [26] of both the ground and excited states, as the multiparty entanglement in the approximate CV surface-code model would decrease in this case (but it would not entirely vanish, because of the intrinsic squeezing of the C_Z gates for CV graph-state generation).

In the Kitaev lattice model, the quasiparticles are particularly well localized: An e particle is on a single star and an m -particle is on a single plaquette. Based on this perfect localization, the proposal for implementing the anyonic model via the simple surface code has been demonstrated

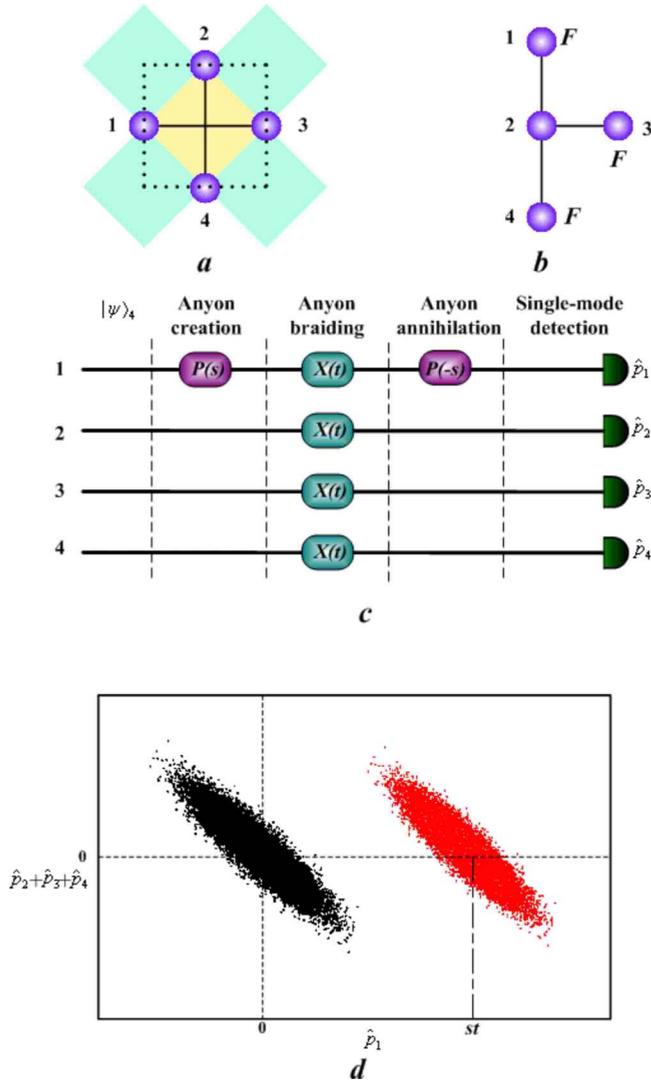


FIG. 2. (Color online) Surface code state with four modes used for demonstration of anyonic statistics. (a) The anyonic ground state $|\psi\rangle_4$, forming one star and four plaquettes, is given by a four-mode GHZ state with the modes labeled 1–4. (b) The four-mode cluster state which after Fourier transforms on modes 1, 3, and 4, is equivalent to the anyonic ground state $|\psi\rangle_4$. (c) The quantum circuit for creation, braiding, and annihilation of the anyons, and for the final state detection. (d) The correlation diagram of the momentum linear combination operator $\hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4$ with (red) and without a closed loop of anyon m around e (black) in the case of finite squeezing.

experimentally with four-photon [14] and six-photon [13] graph states. The four-photon experiment relies upon only four photonic qubits entangled in a four-party GHZ state. This simple system is sufficient for demonstrating a small Kitaev lattice (with one star and four plaquettes) and a non-trivial braiding operation, because the remaining eight qubits of the total 12-qubit system factor out from the four-qubit GHZ state in a fully separable product state [14]. In the CV regime, we propose to pursue an analogous strategy. In fact, four-mode entangled CV GHZ and cluster states have been demonstrated already experimentally [18,19]. In a small-

scale surface-code experiment, it is crucial to verify the full inseparability of the GHZ-type ground state as well as that of the excited state, i.e., a displaced GHZ state. These simple systems may then also be used to implement and detect minimal braiding loops [Figs. 2(a)–2(c)] with four-mode entangled GHZ states. The complete verification requires detecting the position-momentum linear combinations corresponding to the stabilizer set

$$X_1(t)X_2(t)X_3(t)X_4(t),$$

$$Z_1(t)Z_2^{-1}(t), Z_2(t)Z_3^{-1}(t), Z_3(t)Z_4^{-1}(t), \quad (5)$$

including first and second moments [26].

However, for the simple four-mode GHZ states, there is no way to show the robustness of the topological braiding operation when the anyons are moved along different paths. In order to demonstrate such robustness of the anyon operations against different braiding paths, more modes would be needed, for instance, nine-mode states in analogy to the nine-qubit states of Ref. [12]. The corresponding CV nine-mode ground state $|\psi\rangle_9$ is locally equivalent to a CV nine-mode graph state, which can be efficiently generated via nine off-line single-mode squeezed states and arrays of beam splitters [17] instead of the projective measurements on a prepared 2D CV cluster state as described previously [28,29]. In these larger lattices, there are several different loops for the quasi-particles. The anyonic statistics, however, would only depend on the topological character of the loop.

VII. CONCLUSION

In summary, we have proposed a protocol to create, manipulate, and detect anyonic states, including demonstration of their fusion rules and their fractional statistics, in the regime of continuous quantum variables. In our continuous-variable approach to Kitaev's surface code model, the main features of anyons can be obtained via simple Gaussian operations starting from the creation of suitable continuous-variable graph states. With regard to a potential optical realization, this means that all ingredients for a full verification can be implemented in a very efficient way: Graph-state generation only requires off-line squeezing and linear optics; the surface-code ground states are then obtained either through homodyne measurements or via local phase rotations; anyon creation and braiding-dependent state transitions can be also detected through Gaussian operations, i.e., online squeezing transformations (which could be shifted off-line [30]) and homodyne detections (including first and second moments of quadrature combinations in order to verify the fully inseparable ground and excited states and to discriminate them against each other).

This work is a first step towards further theoretical investigations of topological quantum computation over continuous variables and experimental proof-of-principle demonstrations of anyonic, topological features via relatively simple, stable optical systems, as opposed to the more traditional approaches within condensed matter physics.

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